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ON SPECTRAL ANALYSIS OF NON-MONIC MATRIX AND OPERATOR POLYNOMIALS, I. REDUCTION TO MONIC POLYNOMIALS

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ABSTRACT
Three recent papers [1, 2, 3] developed the basic concepts of a spectral theory for matrix and operator monic polynomials. In this paper we continue the study, replacing the requirement of monicness by a weaker condition.

Introduction

This paper is an extension of the theory developed in [1, 2, 3] for monic polynomials. Really we require here that the polynomials have a finite number of spectral points only. This extension leads to complications which make it necessary to consider the spectrum at infinite.

The present paper is the first of three parts. The first two parts are concerned with the finite dimensional case, and the third with the infinite dimensional case. In the second part will be studied the influence of the spectral structure at finite points only.

The basic strategy of this paper is to reduce the problems for non-monic polynomials to problems for monic polynomials and then to apply the results of [1, 2, 3].

§1. General definitions

Let $\mathbf{C}_n$ be the complex linear vector space of dimension n, and let B_n be the algebra of all $n \times n$ matrices with complex entries. Let

$$L(\lambda) = \sum_{j=1}^{m} \lambda^j A_j$$

be a matrix polynomial with $A_j \in B_n$ and argument $\lambda \in \mathbf{C}$. The point $\lambda_0 \in \mathbf{C}$ is

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an eigenvalue of $L(\lambda)$ if $\det L(\lambda_0) = 0$. The set of all eigenvalues of $L(\lambda)$ is called the spectrum of $L(\lambda)$, and denoted by $\sigma(L)$. The polynomial $L(\lambda)$ is regular if $\sigma(L) \neq \mathbf{C}$ or equivalently if $\det L(\lambda) \not\equiv 0$. In this case the spectrum $\sigma(L)$ is either a finite set or else it is empty.

In this paper we deal only with regular polynomials so the word "regular" will be omitted. We suppose also that $L(0) = I$, and call such polynomials comonic. In fact there is no loss of generality because if $L(a)$ is invertible we can shift the argument and study the polynomial $L^{-1}(a) \cdot L(\lambda + a)$ in place of $L(\lambda)$.

Let $L(\lambda)$ be matrix polynomial and let $\lambda_0 \in \sigma(L)$. Then there exists a holomorphic vector valued function $\phi(\lambda)$ with values in $\mathbf{C}_n$ and $\phi(\lambda_0) \neq 0$, such that the function $L(\lambda)\phi(\lambda)$ vanishes at the point λ_0. We call $\phi(\lambda)$ a root function of $L(\lambda)$ corresponding to λ_0. The order of λ_0 as a zero of $L(\lambda)\phi(\lambda)$ is called the multiplicity of $\phi(\lambda)$, and the vector $\phi_0 = \phi(\lambda_0)$ an eigenvector of $L(\lambda)$ corresponding to λ_0. The eigenvectors corresponding to λ_0 form a linear space: $\operatorname{Ker} L(\lambda_0)$. By the rank of an eigenvector ϕ_0 we mean the maximum of the multiplicities of all root functions $\phi(\lambda)$ such that $\phi(\lambda_0) = \phi_0$. The rank of ϕ_0 will be denoted $\operatorname{rank} \phi_0$. A root function $\phi(\lambda)$ such that $\phi(\lambda_0) = \phi_0$ and the multiplicity of $\phi(\lambda)$ is equal to $\operatorname{rank} \phi_0$, is called a maximal root function. Note that if

$$\phi(\lambda) = \sum_{j=0}^{\infty} \phi_j (\lambda - \lambda_0)^j$$

is a maximal root function and if $r = \operatorname{rank} \phi_0$, then the chain of vectors $\phi_0, \phi_1, \cdots, \phi_{r-1}$ is a Jordan chain of $L(\lambda)$ corresponding to the eigenvalue λ_0, i.e.

$$\sum_{i=0}^{j} \frac{1}{i!} L^{(i)}(\lambda_0)\phi_{j-i} = 0, \quad j = 0, \cdots, r-1.$$

The vectors $\phi_1, \cdots, \phi_{r-1}$ are called generalized eigenvectors corresponding to ϕ_0 and λ_0.

For every eigenvalue λ_0 of the matrix polynomial $L(\lambda)$ we define a canonical set of eigenvectors and generalized eigenvectors in the following way: let $\phi_0^{(1)} \in \operatorname{Ker} L(\lambda_0)$ be an eigenvector with maximal rank. Let

$$\phi^{(1)}(\lambda) = \sum_{j=0}^{\infty} \phi_j^{(1)}(\lambda - \lambda_0)^j$$

be a maximal root function such that $\phi^{(1)}(\lambda_0) = \phi_0^{(1)}$. Suppose that the root functions

$$\phi^{(k)}(\lambda) = \sum_{i=0}^{\infty} \phi_i^{(k)}(\lambda - \lambda_0)^i, \quad k = 1, \cdots, j-1,$$

are constructed. Let $\phi_0^{(j)}$ be an eigenvector with maximal rank in some direct complement in $\operatorname{Ker} L(\lambda_0)$ of the linear span of the vectors $\phi_0^{(1)}, \cdots, \phi_0^{(j-1)}$. Let

$$\phi^{(j)}(\lambda) = \sum_{i=0}^{\infty} \phi_i^{(j)}(\lambda - \lambda_0)^i$$

be a maximal root function such that $\phi^{(j)}(\lambda_0) = \phi_0^{(j)}$. By a canonical set of eigenvectors and generalized eigenvectors we mean the ordered set

$$\phi_0^{(1)}, \cdots, \phi_{r_1-1}^{(1)}; \phi_0^{(2)}, \cdots, \phi_{r_2-1}^{(2)}; \cdots; \phi_0^{(k)}, \cdots, \phi_{r_k-1}^{(k)},$$

where $r_j = \operatorname{rank} \phi_0^{(j)}$, $j = 1, \cdots, k$; $k = \dim \operatorname{Ker} L(\lambda_0)$. We write such a canonical set of eigenvectors and generalized eigenvectors in matrix form:

$$X(\lambda_0) = (\phi_0^{(1)} \cdots \phi_{r_1-1}^{(1)} \phi_0^{(2)} \cdots \phi_{r_2-1}^{(2)} \cdots \phi_0^{(k)} \cdots \phi_{r_k-1}^{(k)});$$

$$J(\lambda_0) = \operatorname{diag}(J_1, J_2, \cdots, J_k),$$

where J_i is Jordan cell of size r_i with eigenvalue λ_0, and $\operatorname{diag}(J_1, J_2, \cdots, J_k)$ denotes the square block diagonal matrix whose main diagonal is given by $J_1, J_2, \cdots, J_k$. Hence $X(\lambda_0)$ is an $n \times r$ matrix and $J(\lambda_0)$ is an $r \times r$ matrix, where $r = \sum_{j=1}^{k} r_j$ is the multiplicity of λ_0 as a root of $\det L(\lambda)$.

The pair of matrices $(X(\lambda_0), J(\lambda_0))$ is a canonical pair of $L(\lambda)$ corresponding to λ_0. Taking a canonical pair $(X(\lambda_j), J(\lambda_j))$ for every eigenvalue λ_j of $L(\lambda)$, we define a finite canonical pair (X_F, J_F) of $L(\lambda)$:

$$X_F = (X(\lambda_1)X(\lambda_2) \cdots X(\lambda_p)), \qquad J_F = \operatorname{diag}(J(\lambda_1), J(\lambda_2), \cdots, J(\lambda_p)),$$

where p is the number of different eigenvalues of $L(\lambda)$.

Note that J_F is invertible (since $L(0) = I$). Note also that the pair (X_F, J_F) is not determined uniquely by the polynomial $L(\lambda)$: the description of all finite canonical pairs of $L(\lambda)$ (with fixed J_F) is given by the formula $(X_F U, J_F)$, where (X_F, J_F) is any fixed finite canonical pair of $L(\lambda)$, and U is any invertible matrix which commutes with J_F. The finite canonical pair (X_F, J_F) does not determine $L(\lambda)$ uniquely either: any matrix polynomial of the form $V(\lambda)L(\lambda)$, where $V(\lambda)$ is matrix polynomial with $\det V(\lambda) \equiv \text{const} \neq 0$, has the same finite canonical pairs as $L(\lambda)$. In order to determine $L(\lambda)$ uniquely we have to consider (together with (X_F, J_F)) an additional canonical pair (X_∞, J_∞) of $L(\lambda)$ for $\lambda = \infty$, which is defined below.

Let $\psi_j^{(i)}$, $j = 0, 1, \cdots, s_i - 1$; $i = 1, \cdots, q$ be a canonical set of eigenvectors and generalized eigenvectors of the holomorphic (at infinity) matrix function $\lambda^{-l}L(\lambda)$ corresponding to the eigenvalue $\lambda = \infty$ (where l is the degree of $L(\lambda)$, i.e. the maximal integer j such that $L^{(j)}(\lambda) \neq 0$). We use the following notations:

$$X_\infty = (\psi_0^{(1)} \cdots \psi_{s_1-1}^{(1)} \psi_0^{(2)} \cdots \psi_{s_2-1}^{(2)} \cdots \psi_0^{(q)} \cdots \psi_{s_q-1}^{(q)}),$$

$$J_\infty = \operatorname{diag}(J_{\infty 1}, J_{\infty 2}, \cdots, J_{\infty q});$$

where $J_{\infty j}$ is a nilpotent Jordan cell of size s_j. Note that (X_∞, J_∞) is a canonical pair of the matrix polynomial $\tilde{L}(\lambda) = \lambda^l L(\lambda^{-1})$ corresponding to the eigenvalue $\lambda = 0$.

Now we define a canonical pair (X, J) of $L(\lambda)$ as two matrices $X = (X_F X_\infty)$, $J = \operatorname{diag}(J_F^{-1}, J_\infty)$ of sizes $n \times nl$ and $nl \times nl$.

EXAMPLE 1.1. Consider the matrix polynomial

$$L(\lambda) = \begin{bmatrix} -(\lambda-1)^3 & \lambda \\ 0 & \lambda+1 \end{bmatrix}.$$

In this case we can put down

$$X_F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -8 \end{bmatrix}, \quad J_F = \operatorname{diag}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, -1 \right),$$

and

$$X_\infty = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Therefore a canonical pair for $L(\lambda)$ is given by

$$X = (X_F X_\infty) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -8 & 1 & 0 \end{bmatrix},$$

$$J = \operatorname{diag}(J_F^{-1}, J_\infty) = \operatorname{diag}\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, -1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right).$$

§2. Companion matrix and linearizations

Let

$$L(\lambda) = I + \sum_{j=1}^{l} \lambda^j A_j$$

be a matrix polynomial of degree l. The matrix

$$R = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ -A_l & -A_{l-1} & -A_{l-2} & \cdots & -A_1 \end{bmatrix}$$

is called the companion matrix of $L(\lambda)$. Using the companion matrix, we have two linearizations of $L(\lambda)$.

(a) Linearization in the finite complex plane:

$$I - \lambda R = B(\lambda) \cdot \operatorname{diag}(L(\lambda), I, \cdots, I) \cdot C(\lambda), \tag{1}$$

where

$$B(\lambda) = \begin{bmatrix} 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & I & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & I & \cdots & 0 & 0 \\ I & \lambda A_2 + \lambda^2 A_3 + \cdots + \lambda^{l-1} A_l & \cdots & \lambda A_{l-1} + \lambda^2 A_l & \lambda A_l \end{bmatrix}$$

and

$$C(\lambda) = \begin{bmatrix} 0 & 0 & 0 & \cdots & I \\ & & & \cdots & -\lambda I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & I & -\lambda I & \cdots & 0 \\ I & -\lambda I & 0 & \cdots & 0 \end{bmatrix}$$

are everywhere invertible matrix polynomials. Indeed, by straightforward calculation we obtain

$$C^{-1}(\lambda) = \begin{bmatrix} \lambda^{l-1} I & \lambda^{l-2} I & \cdots & \lambda I & I \\ \lambda^{l-2} I & \lambda^{l-3} I & \cdots & I & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$(I - \lambda R) C^{-1}(\lambda) = B(\lambda) \operatorname{diag}(L(\lambda), I, \cdots, I).$$

(b) Linearization at the point ∞:

$$\lambda^{-1} I - R = E(\lambda^{-1}) \cdot \operatorname{diag}(\lambda^{-l} L(\lambda), I, \cdots, I) \cdot F(\lambda^{-1}), \tag{2}$$

where

$$E(\lambda)=\begin{bmatrix} 0 & 0 & \cdots & I & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & I & \cdots & 0 & \\ I & -(\lambda I+A_1) & \cdots & -(\lambda^{l-1}I+\lambda^{l-2}A_1+\cdots+A_{l-1}) & \end{bmatrix}$$

and

$$F(\lambda)=\begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \lambda I & -I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \lambda I & \cdots & 0 & 0 \\ \lambda I & -I & \cdots & 0 & 0 \end{bmatrix}$$

are everywhere invertible matrix polynomials. Indeed, a straightforward calculation gives

$$F^{-1}(\lambda)=\begin{bmatrix} I & 0 & \cdots & & 0 \\ \lambda I & 0 & \cdots & & -I \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda^{l-1}I & -I & \cdots & -\lambda^{l-3}I & -\lambda^{l-2}I \end{bmatrix}$$

and

$$(\lambda^{-1}I-R)F^{-1}(\lambda^{-1})=E(\lambda^{-1})\operatorname{diag}(\lambda^{-l}L(\lambda),I,\cdots,I).$$

Note that if (X,J) is a canonical pair of $L(\lambda)$ then J and R are similar. We shall prove this later in detail.

§3. Connection with monic polynomials

Throughout this section let $L(\lambda)$ be a comonic matrix polynomial of degree l.

For $m=l,l+1,\cdots$ define a monic polynomial $\tilde{L}_m(\lambda)=\lambda^m L(\lambda^{-1})$ of degree m. (By definition, a matrix polynomial is monic if its leading coefficient is I.) We shall denote the column matrix

$$\begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_p \end{bmatrix}$$

by $\operatorname{col}(T_j)_{j=1}^p$. A pair of matrices $(\tilde{X}_m,\tilde{J}_m)$ (where $\tilde{X}_m$ is an $n\times mn$ matrix and $\tilde{J}_m$

is an $mn \times mn$ matrix) is called a standard pair for the monic polynomial $\tilde{L}_m(\lambda)$ if $\operatorname{col}(\tilde{X}_m\tilde{J}_m^{j-1})_{j=1}^m$ is invertible and if the following representation holds:

$$\tilde{L}_m(\lambda) = \lambda^m I - \tilde{X}_m(\tilde{J}_m)^m(V_1 + V_2\lambda + \cdots + V_m\lambda^{m-1}),$$

where

$$(V_1 V_2 \cdots V_m) = [\operatorname{col}(\tilde{X}_m\tilde{J}_m^{j-1})_{j=1}^m]^{-1}.$$

For more information about standard pairs of monic matrix polynomials we refer to [1].

In this section we find the connection between canonical pairs (X, J) for $L(\lambda)$ and standard pairs $(\tilde{X}_m, \tilde{J}_m)$ for $\tilde{L}_m(\lambda)$.

Consider first the case $m = l$ and denote $\tilde{L}(\lambda) = \tilde{L}_l(\lambda)$. Then the key theorem of this paper is

THEOREM 3.1. *Every canonical pair* (X, J) *for* $L(\lambda)$ *is a standard pair for* $\tilde{L}(\lambda)$.

PROOF. Let $X = (X_F X_\infty)$, let $J = \operatorname{diag}(J_F^{-1}, J_\infty)$, and let $A = (x_0 x_1 \cdots x_r)$ be a Jordan chain for $L(\lambda)$ taken from X_F, and corresponding to the eigenvalue $\lambda \neq 0$. Let

$$K = \begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_0 \end{bmatrix}$$

be a matrix of size $(r+1)\times(r+1)$. The columns of $A \cdot K^{-(l-1)} = \tilde{A} = (\tilde{x}_0\tilde{x}_1 \cdots \tilde{x}_r)$ again from a Jordan chain for $L(\lambda)$, and replacing A by $\tilde{A}$ in X_F, we obtain another canonical set of eigenvectors and generalized eigenvectors.

Let R be the companion matrix of $L(\lambda)$. If the vectors $Y = (y_0 y_1 \cdots y_r)$ are defined by

$$y_i = \sum_{j=0}^{i} \frac{1}{j!}[(C^{-1})^{(j)}(\lambda_0)][\operatorname{col}(\delta_{k1}\tilde{x}_{i-j})_{k=1}^l], \quad i = 0, 1, \cdots, r,$$

then (as follows from (1)) Y is a Jordan chain for $I - \lambda R$ corresponding to the eigenvalue λ_0. Now as

$$C^{-1}(\lambda) = \begin{bmatrix} \lambda^{l-1}I & \lambda^{l-2}I & \cdots & \lambda I & I \\ \lambda^{l-2}I & \lambda^{l-3}I & \cdots & I & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I & 0 & \cdots & 0 & 0 \end{bmatrix}$$

we have: $Y = \operatorname{col}(\tilde{A}K^{l-j-1})_{j=0}^{l-1}$.

It is easy to check that the columns u_j, $j = 0, 1, \cdots, r$ of $YM(\lambda_0)$ (where $M(\lambda_0) = ((-1)^j \binom{j-1}{i-1} \lambda_0^{j+i})_{i,j=0}^r$ and it is assumed that $\binom{-1}{-1} = 1$ and $\binom{p}{q} = 0$ for $q > p$ or $q = -1$ and $p > -1$) form a Jordan chain for $\lambda I - R$ corresponding to the eigenvalue λ_0^{-1}. Using the linearization (2) we obtain that the vectors $t_0, t_1, \cdots, t_r$, where $t_i = F(\lambda_0^{-1})u_i + F'(\lambda_0^{-1})u_{i+1}$, $i = 0, \cdots, r$ (by definition $u_{-1} = 0$), define a Jordan chain for the polynomial $\operatorname{diag}(\tilde{L}(\lambda), I, \cdots, I)$ corresponding to λ_0^{-1}. Taking the first n components of the nl-dimensional vectors t_i, we see that the columns of $\tilde{A}K^{l-1}M(\lambda_0) = AM(\lambda_0)$ form a Jordan chain for $\tilde{L}(\lambda)$ corresponding to λ_0^{-1}.

Applying this construction to every Jordan chain from X_F, we obtain a set of eigenvectors and generalized eigenvectors of $\tilde{L}(\lambda)$ corresponding to the non-zero eigenvalues. It is not hard to see that this set is a canonical.

Let $X_F = (X_1 X_2 \cdots X_s)$ and $J_F = \operatorname{diag}(J_1^{-1}, J_2^{-1}, \cdots, J_s^{-1})$, where X_j is the j-th Jordan chain of X_F, and J_j is the corresponding Jordan matrix with eigenvalue $\lambda_j \neq 0$. Then $\tilde{X} = (X_1 \cdot M(\lambda_1), \cdots, X_s \cdot M(\lambda_s), X_\infty)$ is a canonical set of eigenvectors and generalized eigenvectors of $\tilde{L}(\lambda)$ with corresponding Jordan matrix

$$\tilde{J} = \operatorname{diag}(J_1 + (\lambda_1^{-1} - \lambda_1)I, \cdots, J_s + (\lambda_s^{-1} - \lambda_s)I, J_\infty).$$

Theorem 2 of [4] states that

$$[\tilde{L}(\lambda)]^{-1} = \tilde{X}(\lambda I - \tilde{J})^{-1}\tilde{Y}$$

whenever $\tilde{X}$ is a complete canonical set of eigenvectors and generalized eigenvectors of $\tilde{L}(\lambda)$, and $\tilde{J}$ is the corresponding Jordan matrix. Theorem 14 of [2] then assures that $(\tilde{X}, \tilde{J})$ is a standard pair for $\tilde{L}(\lambda)$.

The identity (which follows from the definition of $M(\lambda_j)$)

$$J_j M(\lambda_j)[J_j + (\lambda_j^{-1} - \lambda_j)I] = M(\lambda_j)$$

implies that $X = \tilde{X}M^{-1}$, $J = M\tilde{J}M^{-1}$, where $M = \operatorname{diag}(M(\lambda_1), \cdots, M(\lambda_s), I)$, and therefore that (X, J) is a standard pair for $\tilde{L}(\lambda)$.

Remark. A Jordan form for $\operatorname{diag}(J_F^{-1}, J_\infty)$ can be included in a canonical pair for $\tilde{L}(\lambda)$. Namely, $(X \cdot \operatorname{diag}(M, I), \tilde{J})$ is a canonical pair for $\tilde{L}(\lambda)$, where $X = (X_F X_\infty)$ and $\tilde{J} = \operatorname{diag}(\tilde{J}_F, J_\infty)$; the matrix $\tilde{J}_F$ is obtained from J_F by replacing λ_j by λ_j^{-1} on the main diagonal of J_F; the matrix M is defined as in the proof of Theorem 3.1. Note that $M\tilde{J}_F M^{-1} = J_F^{-1}$.

Example 3.1. Consider Example 1.1. Then, for this example, the matrix M defined in the proof of Theorem 3.1 is equal to

$$M = \operatorname{diag}\left(1, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, 1\right),$$

and the canonical pair (X, J) of $L(\lambda)$ given by

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -8 & 1 & 0 \end{bmatrix},$$

$$J = \operatorname{diag}\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, -1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

is at the same time a standard pair for the monic polynomial

$$\tilde{L}_3(\lambda) = \lambda^3 L(\lambda^{-1}) = \begin{bmatrix} \lambda^3 - 3\lambda^2 + 3\lambda - 1 & \lambda^2 \\ 0 & \lambda^3 + \lambda^2 \end{bmatrix}.$$

We consider now the case $m > l$. Divide the matrices $\tilde{X}_m$ and $\tilde{J}_m$ into two parts: $\tilde{X}_m = (\tilde{X}_{m0}\tilde{X}_{m1})$, $\tilde{J}_m = \operatorname{diag}(\tilde{J}_{m0}, \tilde{J}_{m1})$, where the subscript 0 denotes the part corresponding to the eigenvalue 0 of $\tilde{L}_m$, and the subscript 1 denotes the part corresponding to the non-zero eigenvalues of $\tilde{L}_m$.

Let $(\tilde{X}_l, \tilde{J}_l)$ be a standard pair for $\tilde{L}_l$ with Jordan matrix $\tilde{J}_{l0} = \operatorname{diag}(J_l^{(1)}, J_l^{(2)}, \cdots, J_l^{(r)})$, where $\tilde{J}_l^{(j)}$ is a nilpotent Jordan block of size K_j, $j = 1, \cdots, r$. Let $\tilde{X}_{l0} = (\tilde{X}_l^{(1)}\tilde{X}_l^{(2)} \cdots \tilde{X}_l^{(r)})$ be the corresponding partition of $\tilde{X}_{l0}$. Then a standard pair $(\tilde{X}_m, \tilde{J}_m)$ for $\tilde{L}_m(\lambda), m > l$, is given by the following formulas:

$$\tilde{X}_{m1} = \tilde{X}_{l1};\ \tilde{J}_{m1} = \tilde{J}_{l1}; \qquad \tilde{J}_{m0} = \operatorname{diag}(\tilde{J}_m^{(1)}, \tilde{J}_m^{(2)}, \cdots, \tilde{J}_m^{(n)}),$$

where $\tilde{J}_m^{(j)}$ is a nilpotent Jordan cell of size $k_j + (m - l)$ (by definition $k_j = 0$ for $j > r$) and

$$\tilde{X}_{m0} = (\tilde{X}_l^{(1)} 0_{m-l} \tilde{X}_l^{(2)} 0_{m-l} \cdots \tilde{X}_l^{(r)} 0_{m-l} \alpha_{r+1} 0_{m-l-1} \cdots \alpha_n 0_{m-l-1}),$$

where 0_k denotes k columns of zeroes, and $\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_n$ is a basis of some direct complement of $\operatorname{Ker} \tilde{L}_l(0)$ in $\mathbf{C}_n$.

Indeed, from the representation $\tilde{L}_m(\lambda) = \lambda^{m-l}\tilde{L}_l(\lambda)$ it is easily seen (from the Smith form) that $\tilde{J}_m$ has the structure mentioned in the statement above. To complete the proof it remains only to notice that $u_0, u_1, \cdots, u_s, 0, \cdots, 0$ ($m - l$ zeroes) is a Jordan chain for $\tilde{L}_m(\lambda)$ for every Jordan chain $u_0, u_1, \cdots, u_s$ for $\tilde{L}_l(\lambda)$ corresponding to the eigenvalue 0, because

$$\frac{1}{j!}\tilde{L}_l^{(j)}(0)=\frac{1}{(j+m-l)!}\tilde{L}_m^{(j+m-l)}(0).$$

§4. Basic pairs

Let

$$L(\lambda)=I+\sum_{i=1}^{l}\lambda^i A_i$$

be matrix polynomial of degree $\leqq l$, and let R be its companion matrix.

A pair (Q,T) of matrices is called a basic pair for $L(\lambda)$ if Q is a $n\times nl$ matrix, T is a $nl\times nl$ matrix, and for some invertible $nl\times nl$ matrix S, $Q=(I0\cdots 0)S$ and $T=S^{-1}RS$. Note that if (Q,T) is a basic pair for $L(\lambda)$ and S is invertible, then $(QS,S^{-1}TS)$ is also a basic pair for $L(\lambda)$.

THEOREM 4.1. *Let (X,J) be a canonical pair for $L(\lambda)$. Then*

$$(\operatorname{col}(XJ^i)_{i=0}^{l-1})J=R(\operatorname{col}(XJ^i)_{i=0}^{l-1}). \tag{3}$$

In particular, J and R are similar, and the pair (X,J) is a basic pair for $L(\lambda)$.

PROOF. The equality (3) is equivalent to

$$XJ^l+A_lX+A_{l-1}XJ+\cdots+A_1XJ^{l-1}=0$$

(since (X,J) is a standard pair of the monic polynomial $\tilde{L}(\lambda)=\lambda^lL(\lambda^{-1})$), and the last equality is proved in [1]. The invertibility of $\operatorname{col}(XJ^i)_{i=0}^{l-1}$ is proved in [1] as well.

This theorem coupled with the results of [1] implies that the following properties are equivalent:

(i) (Q,T) is a basic pair for $L(\lambda)$;

(ii) (Q,T) is a standard pair for $\tilde{L}(\lambda)=\lambda^lL(\lambda^{-1})$;

(iii) $\operatorname{col}(QT^i)_{i=0}^{l-1}$ is invertible and

$$QT^l+A_1QT^{l-1}+\cdots+A_lQ=0.$$

Note that T is the second matrix of a basic pair for $L(\lambda)$ if and only if the following two linearizations hold:

$$\operatorname{diag}(\lambda^{-l}L(\lambda),I,\cdots,I)=G_1(\lambda^{-1})(\lambda^{-1}I-T)G_2(\lambda^{-1}), \tag{4}$$

$$\operatorname{diag}(L(\lambda),I,\cdots,I)=H_1(\lambda)(I-\lambda T)H_2(\lambda), \tag{5}$$

where $G_1^{\pm1}(\lambda)$, $G_2^{\pm1}(\lambda)$, $H_1^{\pm1}(\lambda)$, $H_2^{\pm1}(\lambda)$ are matrix polynomials. The linearization (4) (but generally speaking not (5)) is sufficient to ensure that T is the second matrix of a basic pair for $L(\lambda)$.

To illustrate the theory let us now consider the differential equation

$$L\left(\frac{d}{dt}\right) f(t) = 0 \tag{6}$$

and the difference equation

$$L(\Delta)g_r = 0; \qquad \Delta g_r = g_{r+1}, \quad r = 1, 2, \cdots. \tag{7}$$

Let (Q, T) be a basic pair for $L(\lambda)$, and let Λ be the kernel of the projector

$$P = \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} (\lambda I - T)^{-1} d\lambda,$$

where $\varepsilon > 0$ is small enough. Then the general solution of (6) and (7) is given by the formulas

$$\begin{aligned} f(t) &= Q \cdot \exp[t(T \mid \Lambda)^{-1}]x, \quad x \in \Lambda; \\ g_r &= Q(T \mid \Lambda)^{-r-1}x, \quad x \in \Lambda, \quad r = 1, 2, \cdots. \end{aligned} \tag{8}$$

Indeed, substituting $f(t)$ from (8) into (6), we get

$$\begin{aligned} L\left(\frac{d}{dt}\right) f(t) &= \left(Q \cdot \exp(tT^{-1}) + \sum_{j=1}^{l} A_j Q T^{-j} \cdot \exp(tT^{-1})\right) x \\ &= \left(QT^{l} + \sum_{j=1}^{l} A_j Q T^{l-j}\right) T^{-l} \cdot \exp(tT^{-1})x = 0. \end{aligned}$$

It is easy to see that the solutions $f(t)$ given by (8) form a k-dimensional vector space, where $k = \dim \Lambda$. Note that $\dim \Lambda$ is just the degree of $\det L(\lambda)$, and since the dimension of the solution space of (6) is equal to the degree of $\det L(\lambda)$ (see [4, 5]), every solution of (6) has the form (8). The proof for the equation (7) is similar.

§5. Representation of matrix polynomials

Using Theorems 3.1 and 4.1 and the representation theorems proved in [1, 2] for monic polynomials we are able to find representations for comonic polynomials: let $L(\lambda)$ be a comonic polynomial and let (Q, T) be a basic pair of $L(\lambda)$; then $L(\lambda)$ has the right standard form

$$L(\lambda) = I - QT^{l}(V_1\lambda^{l} + V_2\lambda^{l-1} + \cdots + V_l\lambda),$$

where $(V_1V_2\cdots V_l)=[\mathrm{col}(QT^i)_{i=0}^{l-1}]^{-1}$, and the left standard form

$$L(\lambda)=I-(\lambda^lW_1+\lambda^{l-1}W_2+\cdots+\lambda W_l)T^lY,$$

where $Y=[\mathrm{col}(QT^i)_{j=0}^{l-1}]^{-1}\cdot\mathrm{col}(\delta_{jl}I)_{j=1}^l$ and $\mathrm{col}(W_j)_{j=1}^l=(Y,TY,\cdots,T^{l-1}Y)^{-1}$.

In particular, for each canonical pair (X,J) for $L(\lambda)$, the identity

$$L(\lambda)=I-XJ^l(Z_1\lambda^l+Z_2\lambda^{l-1}+\cdots+Z_l\lambda)$$

holds, where $(Z_1Z_2\cdots Z_l)=[\mathrm{col}(XJ^i)_{i=0}^{l-1}]^{-1}$.

Indeed, since (Q,T) is a standard pair for $\tilde{L}(\lambda)$, the right and left standard forms for $L(\lambda)$ follow from the right and left standard forms for the monic polynomial $\tilde{L}(\lambda)$ (theorems 1 and 3 of [1]). The representation using the canonical pair (X,J) then follows from Theorem 3.1.

THEOREM 5.1. *Let $L_1(\lambda)$ and $L_2(\lambda)$ be comonic matrix polynomials each of which is of degree $\leqq l$. Let $(X_j,J_j)=((X_{jF}X_{j\infty}),\mathrm{diag}(J_{jF}^{-1},J_{j\infty}))$, $j=1,2$, be a canonical pair for $L_j(\lambda)$ which is partitioned into finite and infinite parts.*

If $X_{1F}=X_{2F}$ and $J_{1F}=J_{2F}$, then the quotient of the division of $L_1(\lambda)$ by $L_2(\lambda)$ on the right is an everywhere invertible matrix polynomial of degree $\leqq\max(\gamma,l)+1$, where γ is the smallest positive integer such that $J_{1\infty}^\gamma=J_{2\infty}^\gamma=0$.

PROOF. We divide $L_1(\lambda)$ by $L_2(\lambda)$ on the right using the right standard form as described above. The process of division is similar to that which is described in theorem 6 of [1].

We have:

$$L_j(\lambda)=I-X_jJ_j^l(V_{1j}\lambda^l+V_{2j}\lambda^{l-1}+\cdots+V_{lj}\lambda),\quad j=1,2,$$

where $(V_{1j}V_{2j}\cdots V_{lj})=[\mathrm{col}(X_jJ_j^{i-1})_{i=1}^l]^{-1}$.

Let

$$F_{\alpha\beta}=X_1J_1^\alpha V_{\beta 1},\quad G_{\alpha\beta}=X_2J_2^\alpha V_{\beta 2},\quad \beta=1,2,\cdots,l;\quad \alpha=0,1,2,\cdots,$$

and

$$\Phi(k,i)=G_{l+i,l-k}-F_{l,l-i-k}-\sum_{j=0}^{i-1}F_{l,l-j}G_{l+i-1-j,l-k},$$

$k=0,1,\cdots,l-1$; $i=0,1,2,\cdots$, with the understanding that $G_{pi}=F_{pi}=0$ for $i\leqq 0$. Then

$$L_1(\lambda)=\left[I+\sum_{j=0}^{i}\Phi(0,j)\lambda^{j+1}\right]L_2(\lambda)+\sum_{j=0}^{l-1}\Phi(j,i)\lambda^{i+j+2}.$$

Hence it is sufficient to prove that $\Phi(k,i)=0$ for $i\geqq\max(\gamma,l)$.

By the definition of γ it follows that $X_1J_1^\mu = X_2J_2^\mu$ for every $\mu \geqq \gamma$, since $X_{1F} = X_{2F}$, $J_{1F} = J_{2F}$. Moreover, $F_{l,l-i-k} = 0$ for $k = 0, 1, \cdots, l-1$ and $i \geqq l$ (because then $l - i - k \leqq 0$). Therefore, under the assumption $i \geqq \max(\gamma, l)$ we have:

$$\begin{aligned}\Phi(k,i) &= G_{l+i,l-k} - \sum_{j=0}^{i-1} F_{l,l-j}G_{l+i-1-j,i-k}\\ &= X_2J_2^{l+i}V_{l-k,2} - \sum_{j=0}^{l-1} X_1J_1V_{l-j,1}\cdot X_2J_2^{l+i-1-j}V_{l-k,2}\\ &= X_2J_2^{l+i}V_{l-k,2} - X_1J_1^l\left(\sum_{j=0}^{l-1} V_{l-j,1}X_1J_1^{l-1-j}\right)J_1^iV_{l-k,2}\\ &= X_2J_2^{l+i}V_{l-k,2} - X_1J_1^{l+i}V_{l-k,2} = 0.\end{aligned}$$

We are now able to solve the inverse problem, i.e. we are able to reconstruct a matrix polynomial given its finite and infinite Jordan chains.

A pair of matrices (Y, K) is called an admissible pair if Y is an $n \times p$ matrix and K is a $p \times p$ matrix.

THEOREM 5.2. *Let (Y, K) and (W, K_0) be admissible pairs with an invertible Jordan matrix K and a nilpotent Jordan matrix K_0 which are such that the matrix $\operatorname{col}(YK^j, WK_0^{l-1-j})_{j=0}^{l-1}$ is square and invertible for some integer $l > 0$. Then there exists a unique comonic matrix polynomial $L(\lambda)$ of degree $\leqq l$ for which the pair $((YW), \operatorname{diag}(K^{-1}, K_0))$ is canonical.*

PROOF. The desired polynomial $L(\lambda)$ is defined by $L(\lambda) = \lambda^l\tilde{L}(\lambda^{-1})$, where $\tilde{L}(\lambda)$ is the monic polynomial of degree l represented by the standard pair $((YW), \operatorname{diag}(K^{-1}, K_0))$. This is easy to check using Theorem 3.1. The uniqueness of $L(\lambda)$ follows from its right standard form.

§6. General triples

We begin with the resolvent representation of the matrix polynomial $L(\lambda)$.

THEOREM 6.1. *Let $L(\lambda)$ be a comonic matrix polynomial of degree $\leqq l$, and let (X, J) be its canonical pair. Then*

$$L^{-1}(\lambda) = P_l^TY(I - \lambda J)^{-1}Y^{-1}P_l$$

where $P_l^T = (0 \cdots 0\, I)$, $Y = \operatorname{col}(XJ^j)_{j=0}^{l-1}$.

PROOF. (X, J) is a standard pair for

$$\tilde{L}(\lambda) = \lambda^l L(\lambda^{-1}),$$

therefore

$$[\tilde{L}(\lambda)]^{-1} = X(\lambda I - J)^{-1} Z,$$

where

$$Z = [\operatorname{col}(XJ^i)_{i=0}^{l-1}]^{-1} \cdot \operatorname{col}(\delta_{jl} I)_{j=1}^{l}$$

(see corollary 1 of [2]). Using this equation and the biorthogonality condition $XJ^jZ = 0$ for $j = 0, \cdots, l-2$, $XJ^{l-1}Z = 1$, we see that, for λ close enough to zero,

$$\begin{aligned} L^{-1}(\lambda) &= \lambda^{-l}[\tilde{L}(\lambda^{-1})]^{-1} = \lambda^{-l+1} X(I - \lambda J)^{-1} Z \\ &= \lambda^{-l+1} X(I + \lambda J + \lambda^2 J^2 + \cdots) Z \\ &= \lambda^{-l+1} X(\lambda^{l-1} J^{l-1} + \lambda^l J^l + \cdots) Z = XJ^{l-1}(I - \lambda J)^{-1} Z \end{aligned}$$

and the theorem follows.

A triple of matrices (Q, T, B) (where Q is an $n \times p$ matrix, T is a $p \times p$ matrix and B is a $p \times n$ matrix) is called a general triple for $L(\lambda)$ if $L^{-1}(\lambda) = Q(I - \lambda T)^{-1} B$. The integer p is called the order of the general triple (Q, T, B).

Let (Q, T) be a basic pair for $L(\lambda)$. Then (Q, T, B) with $B = T^{l-1}[\operatorname{col}(QT^i)_{i=0}^{l-1}]^{-1} \cdot \operatorname{col}(\delta_{jl} I)_{j=1}^{l}$ is a general triple for $L(\lambda)$. It is possible to check that by repeating the proof of Theorem 6.1. We shall see that not every general triple is such an extension of a basic pair.

As before, define the monic polynomial $\tilde{L}(\lambda) = \lambda^l L(\lambda^{-1})$. A triple $(\tilde{Q}, \tilde{T}, \tilde{B})$ is called a standard triple for $\tilde{L}(\lambda)$, if $(\tilde{Q}, \tilde{T})$ is a standard pair for $\tilde{L}(\lambda)$ and

$$\tilde{B} = [\operatorname{col}(\tilde{Q}\tilde{T}^{j-1})_{j=1}^{l}]^{-1} \cdot \operatorname{col}(\delta_{jl} I)_{j=1}^{l}.$$

The following lemma gives a partial description of general triples.

Lemma 6.1. *Let $(\tilde{Q}, \tilde{T}, \tilde{B})$ be a standard triple for $\tilde{L}(\lambda)$. Let X and Y be square matrices such that $X\tilde{T}^j Y = \tilde{T}^{j+l-1}$, $j = 0, 1, 2, \cdots$. Then $(\tilde{Q}X, \tilde{T}, Y\tilde{B})$ is a general triple for $L(\lambda)$.*

Proof. Expanding

$$L^{-1}(\lambda) = \lambda^{-l}[\tilde{L}(\lambda^{-1})]^{-1} = \tilde{Q} \cdot \lambda^{-l+1}(I - \lambda \tilde{T})^{-1} \tilde{B}$$

as power series in λ for small λ, and using the biorthogonality condition $\tilde{Q}\tilde{T}^j\tilde{B} = 0$, $j = 0, 1, \cdots, l-2$, one can easily check that this expression coincides with $\tilde{Q}X(I - \lambda \tilde{T})^{-1} Y\tilde{B}$.

Taking $X = I$ in Lemma 6.1, we obtain exactly all the basic pairs $(\tilde{Q}, \tilde{T})$ of $L(\lambda)$. It is not hard to show that if $\tilde{T}$ is invertible, then every general triple has the form described in Lemma 6.1 with $X = I$ and $Y = \tilde{T}^{l-1}$, and therefore is an extension of a basic pair for $L(\lambda)$. It is not true for singular $\tilde{T}$ as shows the following example.

EXAMPLE 6.1. Let $L(\lambda) = I$ be a 2×2 matrix polynomial and let $l = 2$ (i.e. we regard $L(\lambda)$ as polynomial $I + \lambda \cdot 0 + \lambda^2 \cdot 0$). The monic polynomial $\tilde{L}(\lambda) = \lambda^2 I$ has a standard triple $(\tilde{Q}, \tilde{T}, \tilde{B})$ with

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \tilde{T} = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right),$$

$$\tilde{B} = \operatorname{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

A triple (Q, T, B) is a general triple for $L(\lambda)$ if and only if $QT^jB = 0$, $j > 0$, $QB = I$. So the triple $Q = (I\ 0)$, $T = 0$, $B = \operatorname{col}(I, 0)$ is a general triple for $L(\lambda)$ which is not an extension of a basic pair (because T is not similar to $\tilde{T}$).

Note that it is possible to replace T by $\left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ in the general triple (Q, T, B). Note also that the general triple (Q, T, B) satisfies the equation

$$QT^l + A_1 QT^{l-1} + \cdots + A_l Q = 0.$$

However the knowledge of a general triple of a matrix polynomial allows us to calculate the coefficients of the polynomial and to find a general triple of a product. These properties which will be proved below partially justify the definition of a general tiple.

THOEREM 6.2. *Let*

$$L(\lambda) = I + \sum_{j=1}^{l} \lambda^j A_j$$

be a matrix polynomial and let (Q, T, B) be its general triple. Then the A_j are determined by the recursive formulas:

$$A_1 = -QTB; \quad A_j = -QT^jB - \sum_{i=1}^{j-1} A_i QT^{j-i}B, \quad j = 2, \cdots, l.$$

PROOF. Write

$$(I + \lambda A_1 + \cdots + \lambda^l A_l) Q (I + \lambda T + \lambda^2 T^2 + \cdots) B = I$$

for λ close enough to zero. Comparing the coefficients of λ^j, $j = 1,2,\cdots,l$ on both sides and using the equality $QB = I$, we obtain the desired formulas.

THEOREM 6.3. *Let $L_1(\lambda)$, $L_2(\lambda)$ be comonic matrix polynomials having general triples (Q_1, T_1, B_1) and (Q_2, T_2, B_2) respectively. Define matrices Q, T, B by*

$$Q = (Q,0), \quad T = \begin{bmatrix} T_1 & B_1 \cdot Q_2 \\ 0 & T_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ T_2 \cdot B_2 \end{bmatrix}.$$

Then (Q, T, R) is a general triple of the product $L_2(\lambda)L_1(\lambda)$.

PROOF. The proof is similar to the proof of theorem 5 of [1]. We have:

$$(I - \lambda T)^{-1} = \begin{bmatrix} (I - \lambda T_1)^{-1} & \lambda (I - \lambda T_1)^{-1} B_1 Q_2 (I - \lambda T_2)^{-1} \\ 0 & (I - \lambda T_2)^{-1} \end{bmatrix}$$

(this can be seen writing $(I - \lambda T)(I - \lambda T)^{-1} = I$). Then

$$Q(I - \lambda T)^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \lambda Q_1 (I - \lambda T_1)^{-1} B_1 Q_2 (I - \lambda T_2)^{-1} B_2$$

$$= \lambda L_1^{-1}(\lambda) L_2^{-1}(\lambda).$$

On the other hand, for λ close enough to zero,

$$Q(I - \lambda T)^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \sum_{j=0}^{\infty} \lambda^j Q T^j \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$= \lambda \cdot \sum_{j=0}^{\infty} \lambda^j Q T^j B = \lambda Q (I - \lambda T)^{-1} B.$$

By comparison with the preceding formula, the result follows.

§7. Divisors of a matrix polynomial

In this section we give a description of all the comonic divisors of a given comonic polynomial $L(\lambda)$ of degree l.

For two given positive integers k_1 and k_2, let $\mathrm{Div}(k_1, k_2)$ be the set of all right comonic divisors $M(\lambda)$ of degree $\leqq k_1$ such that the quotient $L(\lambda)M^{-1}(\lambda)$ has degree $\leqq k_2$. This definition makes sense only if $k_1 + k_2 \geqq l$. As before define the matrix polynomial $\tilde{L}_m(\lambda) = \lambda^m L(\lambda^{-1})$ for integer $m \geqq l$.

We say that the two right divisors $M_1(\lambda)$ and $M_2(\lambda)$ belong to the same class if

$M_1(\lambda) = V(\lambda)M_2(\lambda)$ for some matrix polynomial $V(\lambda)$ which is invertible for all $\lambda \in \mathbf{C}$. According to Theorem 5.1, two divisors belong to the same class if and only if they have canonical pairs with the same finite parts.

Now fix two positive integers k_1 and k_2 such that $k_1 + k_2 \geqq l$ and let $m = k_1 + k_2$. Let (X, J) be a canonical pair for $L(\lambda)$. Then (as in §3) the pair (X, J) can be extended to a standard pair (X_m, J_m) for the monic polynomial $\tilde{L}_m(\lambda)$. It is clear that $M(\lambda) \in \mathrm{Div}(k_1, k_2)$ if and only if $\tilde{M}(\lambda) = \lambda^{k_1} M(\lambda^{-1})$ is a right monic divisor of the polynomial $\tilde{L}_m(\lambda)$. According to [1], each monic divisor $\tilde{M}(\lambda)$ is defined by an invariant subspace Λ of J_m for which the matrix $\mathrm{col}(X_m(J_m \mid \Lambda)^j)_{j=0}^{k_1-1}$ is invertible:

$$M(\lambda) = \lambda^{k_1} I - X_m(J_m \mid \Lambda)^{k_1} \cdot \sum_{j=0}^{k_1-1} \lambda^j V_j,$$

where $[\mathrm{col}(X_m(J_m \mid \Lambda)^j)_{j=0}^{k_1-1}]^{-1} = (V_0 V_1 \cdots V_{k_1-1})$. Conversely, each such invariant subspace Λ of J_m is defined by a monic divisor of degree k_1 of $\tilde{L}_m(\lambda)$. The subspace Λ is called the supporting subspace of $\tilde{M}(\lambda)$. We shall say that Λ is also the supporting subspace of the right divisor

$$M(\lambda) = I - X_m(J_m \mid \Lambda)^{k_1} \cdot \sum_{j=0}^{k_1-1} \lambda^{k_1-j} V_j$$

of $L(\lambda)$.

We now describe the classes of divisors of $L(\lambda)$ in terms of the supporting subspaces. Let Ω be the kernel of the projector

$$P = \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} (\lambda I - J_m)^{-1} d\lambda$$

where $\varepsilon > 0$ is small enough. For every invariant subspace Λ of J_m, the intersection $\Lambda \cap \Omega$ will be called the Ω-projection of Λ. A pair of divisors $M(\lambda), N(\lambda) \in \mathrm{Div}(k_1, k_2)$ belongs to the same class if and only if the Ω-projections of their supporting subspaces coincide. Indeed, define $\tilde{M}(\lambda) = \lambda^{k_1} M(\lambda^{-1})$ and $\tilde{N}(\lambda) = \lambda^{k_1} N(\lambda^{-1})$. It follows then from the proof of Theorem 3.1 that $M(\lambda)$ and $N(\lambda)$ belong to the same class if and only if $\tilde{M}(\lambda)$ and $\tilde{N}(\lambda)$ have the same non-zero eigenvalues and the same corresponding Jordan chains. The last condition is equivalent to the coincidence of the projections of the corresponding supporting subspaces, because Ω is the maximal invariant subspace of J_m such that $J_m \mid \Omega$ is invertible.

The following example shows that the set $\mathrm{Div}(k_1, k_2)$ does not always contain a representative from every class of divisors.

EXAMPLE 7.1. Let $L(\lambda)=\operatorname{diag}((\lambda-1)^2,(2\lambda-1)^2)$. Then it is easy to check that $\operatorname{Div}(1,1)$ contains only one divisor: $\operatorname{diag}(1-\lambda,1-2\lambda)$. But the right comonic divisor $\operatorname{diag}(1,(2\lambda-1)^2)$ of $L(\lambda)$ does not belong to the class of divisors $\operatorname{Div}(1,1)$.

For k_1+k_2 large enough, the set $\operatorname{Div}(k_1,k_2)$ contains a representative from every class of divisors. This statement can be easily proved by considering the Smith form. That will be analysed in more detail in the next part of this paper.

§8. Decomposition into linear factors

Our final result concerns polynomials for which all the elementary divisors are linear.

Let $L(\lambda)$ be a comonic matrix polynomial of degree l, and let (X,J) be its canonical pair. If J can be diagonalized, then $L(\lambda)$ can be decomposed into a product of l linear factors. Indeed, denoting $\tilde{L}(\lambda)=\lambda^l L(\lambda^{-1})$ we have the following decomposition:

$$\tilde{L}(\lambda)=\prod_{j=1}^{l}(\lambda I+Z_j)$$

for some matrices $Z_1,Z_2,\cdots,Z_l$. This is a corollary of theorem 12 of [1]. Therefore,

$$L(\lambda)=\prod_{j=1}^{l}(I+\lambda Z_j).$$

This in turn implies the following result: if the degree of $\det L(\lambda)$ is $nl-1$, and all the elementary divisors of $L(\lambda)$ are linear, then $L(\lambda)$ admits a decomposition into a product of l linear factors.

Note that there exist comonic matrix polynomials which do not admit any decomposition into a product of linear factors; $\left[\begin{smallmatrix}1 & \lambda^2\\ 0 & 1\end{smallmatrix}\right]$ serves as an example.

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